Notes for Exam #1

General Comments

Justify your steps! Are you using a theorem, the hypothesis, something you learned in linear algebra, or from calculus?

Pay careful attention to where elements live. Does the element you are talking about have the right form?

Make sure you can answer these questions no matter what goes in the blanks:

1. What do I have to do to show that a binary operation $*$ is __________? (associative, commutative, etc.)

2. What do I have to do to show that $*$ is not a binary operation? It might not be __________.

3. What do I have to do to show that a map $\phi$ is __________? (one-to-one, onto, etc.)

4. What does a general element of __________ $(\mathbb{Q}, \mathbb{C}, 2\mathbb{Z}, \text{etc})$ look like?

5. How do you check when an element belongs to __________ $(\mathbb{Q}, \mathbb{C}, 2\mathbb{Z}, \text{etc})$?

Section 1: Introduction and Examples

1. Be able to represent and manipulate complex numbers as exponentials, it involves Euler’s formula.

2. We introduced the homomorphism property here.

3. Our first example of an isomorphism was $\langle \mathbb{R}_{2\pi}, +_{2\pi} \rangle \cong \langle U, \cdot \rangle$.

4. Our second example was $\langle \mathbb{Z}_n, +_n \rangle \cong \langle U_n, \cdot \rangle$, where $U_n$ is the $n$th roots of unity, in particular the case $n = 4$.

5. If two sets have the same algebraic properties, it can be easier to use an isomorphism to transform a given equation into an equivalent equation in the other set, where it is easier to solve.

Section 2: Binary Operations

1. Be able to identify a (commutative, associative) binary operation.

2. Binary operators can be defined using tables as well.
3. Read the words of warning on page 24.

4. We also learned about idempotents in the exercises.

Section 3: Isomorphic Binary Structures

1. Intuitively, two binary structures are isomorphic if they are algebraically the same, ie. they only differ by the names given to their elements.

2. Decide if a given function between two binary structures is an isomorphism. Even if its not, the structures may still be isomorphic (the same).

3. Be able to carry out the four step process to show two binary structures are isomorphic.

4. Make sure you know what is a structural property, and what is not a structural property. Refer to the table on page 32.

5. Be able to show two binary structures are not isomorphic.

6. The identity element $e$ is unique, and having an identity element is a structural property. Notice identities map to identities under an isomorphism. (Theorem $\pi$)

Section 4: Groups

1. A group is closed under $*$, and it satisfies axioms $G_1, G_2, G_3$. Be able to check for this.

2. A group is abelian if $*$ is commutative. How can you check if the group is presented as a multiplication table?

3. The left and right cancellation laws are stolen from solving equations in $\mathbb{R}$. In fact, they guarantee that the equations $a * x = b$ and $y * a = b$ have unique solutions.

4. The inverse of $a * b$ mimics the linear algebra rule for $(AB)^{-1}$, $A$ and $B$ matrices.

5. Having a left identity and left inverses is sufficient to define a group. The same is true for a right identity and right inverses. However, having one of each is not enough to guarantee a group (see exercise #30).

6. Identities and inverses place restrictions on the elements of a group table.

Section 5: Subgroups

1. Be careful with notation. We will now abbreviate $a * b$ as $ab$, even if the binary operator is not multiplication.

2. What is the order of a group?

3. Know the difference between proper and improper, trivial and nontrivial subgroups.

4. A subset $H$ is a subgroup if it satisfies the three conditions of Theorem $(\pi + 2)$. Existence is not the issue, do the elements live in the right place?
5. Again, you do not have to reinvent the wheel by verifying the group axioms hold for $H$. What you do have to show is closure in $H$, and that the identity and inverses stay in $H$.

6. Every cyclic subgroup $H$ has a generator $a$ so that $\langle a \rangle = H$. This is also the smallest subgroup of $G$ that contains $a$.

7. You can have a finite cyclic subgroup of an infinite non-cyclic group, for example $\langle i \rangle = U_4 \leq \mathbb{C}$. 